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Foldable cubical complexes of nonpositive curvature

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Abstract We study finite foldable cubical complexes of nonpositive curvature (in the sense of A.D. Alexandrov). We show that such a complex X admits a graph of spaces decomposition. It is also shown that when $\dim X = 3$, X contains a closed rank one geodesic in the 1-skeleton unless the universal cover of X is isometric to the product of two CAT(0) cubical complexes.

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1 Introduction

We study finite foldable cubical complexes with nonpositive curvature (in the sense of A.D. Alexandrov), including the rank rigidity problem of such complexes. Foldable cubical complexes have been studied by W. Ballmann and J. Swiatkowski in [BSw]. Our notion of foldable cubical complexes is slightly more general than that of [BSw] since we do not require gallery connectedness. D. Wise has also studied a class of 2-dimensional cubical complexes ($\mathcal{V}H$-complexes in [W]) which are closely related to foldable cubical complexes.

A *cubical complex* X is a CW-complex formed by gluing unit Euclidean cubes together along faces via isometries. We require that all the cubes inject into X and the intersection of the images of two cubes is either empty or equals the image of a cube. The image of a k-dimensional unit Euclidean cube in X is called a k-cube, and a 1-cube is also called an edge. Let d be the path pseudometric on X. When X is finite dimensional, d is actually a metric and turns X into a complete geodesic space, see [Br].

Let X be a cubical complex. X is called *dimensionally homogeneous* if there is an integer $n \geq 1$ such that each cube of X is a face of some n-cube. A *folding* of X is a combinatorial map $f : X \to C$ onto an n-cube C such that the

restriction of f on each cube is injective. X is *foldable* if it admits a folding. X has *nonpositive curvature* in the sense of A.D. Alexandrov if and only if all the vertex links are flag complexes [BH]. Recall that a simplicial complex is a flag complex if any finite set of vertices that is pairwise joined by edges spans a simplex. By a *FCC* we mean a connected foldable cubical complex that is dimensionally homogeneous, geodesically complete (definition given below) and has nonpositive curvature.

Our first observation about FCCs is the following:

Proposition 2.6 *Let X be a* FCC *of dimension n. Then X admits the structure of a graph of spaces, where all the vertex and edge spaces are $(n-1)$-dimensional* FCCs *and the maps from edge spaces to vertex spaces are combinatorial immersions.*

Notice Proposition 2.6 offers the potential for proving statements about FCCs by inducting on dimension.

We shall give two applications of the above observation. The first concerns the rank rigidity problem for CAT(0) spaces. Let X be a metric space with nonpositive curvature. A curve $\sigma : I \to X$ is a *geodesic* if it has constant speed and is locally distance-minimizing. We say X is *geodesically complete* if every geodesic $\sigma : I \to X$ can be extended to a geodesic $\tilde{\sigma} : \mathbb{R} \to X$. A CAT(0) *space* is a simply connected complete geodesic space with nonpositive curvature. Let Y be a geodesically complete CAT(0) space. We say Y has higher rank if each geodesic $\sigma : \mathbb{R} \to Y$ is contained in a flat plane (the image of an isometric embedding $\mathbb{R}^2 \to Y$). Otherwise we say Y has rank one. There are two main classes of higher rank CAT(0) spaces: symmetric spaces and Euclidean buildings. The following conjecture is still open ([BBu], [BBr2]):

Rank Rigidity Conjecture *Let Y be a geodesically complete* CAT(0) *space and Γ a group of isometries of Y whose limit set is the entire ideal boundary of Y.*

(1) *If Y has higher rank, then Y isometrically splits unless Y is a symmetric space or an Euclidean building;*

(2) *If Y has rank one, then Y contains a periodic rank one geodesic.*

Recall a complete geodesic $\sigma : \mathbb{R} \to Y$ is a *periodic rank one geodesic* if σ does not bound a flat half-plane and there is some $\gamma \in \Gamma$ and $c > 0$ such that $\gamma(\sigma(t)) = \sigma(t+c)$ for all $t \in \mathbb{R}$. The conjecture holds if the action of Γ on Y is

proper and cocompact and Y is a Hadamard manifold [B] or a 2-dimensional polyhedron with a piecewise smooth metric [BBr]. Claim (1) of the conjecture also holds when Y is a 3-dimensional piecewise Euclidean polyhedron and Γ acts on Y properly and cocompactly [BBr2].

Theorem 3.13 *Let X be a finite* FCC *of dimension* 3 *with universal cover $\tilde{X}$ and group of deck transformations Γ.*

(1) *If $\tilde{X}$ has higher rank, then $\tilde{X}$ is isometric to the product of two* CAT(0) FCC*s .*

(2) *If $\tilde{X}$ has rank one, then $\tilde{X}$ contains a periodic rank one geodesic in the* 1*-skeleton.*

As the second application, we address the Tits alternative question for finite FCCs . The result has been established by Ballmann and Swiatkowski [BSw]. We give a new and very short proof.

Theorem 4.1 *Let X be a finite* FCC *. Then any subgroup of $\pi_1(X)$ either contains a free group of rank two or is virtually free abelian.*

The paper is organized as follows. In Section 2 we recall some basic facts about FCCs and show that a FCC has a graph of spaces structure. In Section 3 the rank rigidity problem for 3-dimensional FCCs is discussed. In Section 4 we give a new proof of the Tits alternative for FCCs.

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2 Foldable cubical complexes

2.1 Locally convex subcomplexes

In this section we show that a FCC has many locally convex subcomplexes. The results in this section are more or less known (see [BSw], [C], [DJS]). We include the proofs only for completeness.

For any locally finite piecewise Euclidean complex Y and any $y \in Y$, the link $\mathrm{Link}(Y, y)$ is piecewise spherical. We let d_y be the induced path metric on $\mathrm{Link}(Y, y)$. If Y has nonpositive curvature, then $\mathrm{Link}(Y, y)$ is a CAT(1) space. Let Y be a locally finite piecewise Euclidean complex with nonpositive

curvature and $Z \subset Y$ a subcomplex. For $z \in Z$, a subset $L(Z, z) \subset \mathrm{Link}(Y, z)$ is defined as follows: a point $\xi \in \mathrm{Link}(Y, z)$ belongs to $L(Z, z)$ if there is a nontrivial geodesic segment zx of Y with $zx \subset Z$ such that the initial direction of zx at z is ξ.

For any metric space Z, the *Euclidean cone* over Z is the metric space $C(Z)$ defined as follows. As a set $C(Z) = Z \times [0, \infty)/Z \times \{0\}$. The image of (z, t) is denoted by tz. $d(t_1 z_1, t_2 z_2) = t_1 + t_2$ if $d(z_1, z_2) \geq \pi$, and $d(t_1 z_1, t_2 z_2) = \sqrt{t_1^2 + t_2^2 - 2t_1 t_2 \cos(d(z_1, z_2))}$ if $d(z_1, z_2) \leq \pi$. The point $O := Z \times \{0\}$ is called the cone point of $C(Z)$.

Recall a subset $A \subset M$ of a CAT(1) metric space M is π-convex if for any $a, b \in A$ with $d(a, b) < \pi$ the geodesic segment ab lies in A.

Lemma 2.1 *Let Y be a locally finite piecewise Euclidean complex with non-positive curvature and $Z \subset Y$ a subcomplex. Then Z is locally convex in Y if and only if for each $z \in Z$, $L(Z, z)$ is π-convex in* $\mathrm{Link}(Y, z)$.

Proof For any $y \in Y$ let $C(\mathrm{Link}(Y, y))$ be the Euclidean cone over $\mathrm{Link}(Y, y)$ and O the cone point. For any $r > 0$ let $\bar{B}(y, r) \subset Y$ and $\bar{B}(O, r) \subset C(\mathrm{Link}(Y, y))$ be the closed metric balls with radius r. For any subset $A \subset \mathrm{Link}(Y, y)$, let $C_r(A) \subset C(\mathrm{Link}(Y, y))$ be the subset consisting of points of the form ta, $t \leq r$ and $a \in A$.

Since Y is a locally finite piecewise Euclidean complex and $Z \subset Y$ is a subcomplex, for each $z \in Z$ there is some $r > 0$ and an isometry $g : \bar{B}(z, r) \to \bar{B}(O, r)$ such that $g(\bar{B}(z, r) \cap Z) = C_r(L(Z, z))$. Now it is easy to see that $\bar{B}(z, r) \cap Z$ is convex in $\bar{B}(z, r)$ if and only if $L(Z, z)$ is π-convex in $\mathrm{Link}(Y, z)$. ☐

Let X be a FCC and $f : X \to C$ a fixed folding. Two edges e_1 and e_2 of X are equivalent if $f(e_1)$ and $f(e_2)$ are parallel in C. Let $E_1, \cdots, E_n$ be the equivalence classes of the edges of X. For each nonempty subset $T \subset \{1, 2, \cdots, n\}$ we define a subcomplex X_T of X as follows: a k-cube belongs to X_T if all its edges belong to $E_T := \cup_{i \in T} E_i$. X_T is disconnected in general. We shall see that each component of X_T is locally convex in X.

To prove that the components of X_T are locally convex in X we also need the following lemma. Recall a spherical simplex is all right if all its edges have length $\pi/2$, and a piecewise spherical simplicial complex K is all right if all its simplices are all right. When K is finite dimensional, the path pseudometric on K is a metric that makes K a complete geodesic space [Br].

Lemma 2.2 ([BH], p.211) *Let K be a finite dimensional all right spherical complex and $v \in K$ a vertex. If $\sigma : [a,b] \to K$ is a geodesic such that $\sigma(a), \sigma(b) \notin B(v, \pi/2)$, then each component of $B(v,\pi/2) \cap \sigma$ has length π.*

A dimensionally homogeneous cubical complex Z of dimension n *has no boundary* if each $(n-1)$-cube is contained in the boundaries of at least two n-cubes. Similarly, a dimensionally homogeneous simplicial complex Z of dimension n has no boundary if each $(n-1)$-simplex is contained in the boundaries of at least two n-simplices.

The following proposition is a consequence of Proposition 1.7.1 of [DJS]. It also follows from Lemmas 1.1 and 1.3 of [C]. In addition, W. Ballmann and J. Swiatkowski have made the same observation([BSw], Lemma 3.2(4) and the paragraph at the bottom of p.69 and the top of p.70).

Proposition 2.3 *Let X be a* FCC *and $f : X \to C$ a fixed folding onto an n-cube. Then for any nonempty $T \subset \{1, 2 \cdots, n\}$, each component of X_T is locally convex in X. Furthermore, each component of X_T is also a* FCC .

Proof Let Z be a component of X_T. For any $z \in Z$ we need to show that $L(Z,z) \subset \text{Link}(X,z)$ is π-convex. First assume z is a vertex. Then $\text{Link}(X,z)$ is an all right flag complex. By the definition of X_T we see $L(Z,z)$ is a full subcomplex of $\text{Link}(X,z)$, that is, a simplex of $\text{Link}(X,z)$ lies in $L(Z,z)$ if and only if all its vertices lie in $L(Z,z)$. Let $\xi, \eta \in L(Z,z)$ with $d_z(\xi,\eta) < \pi$. Assume $\xi\eta \not\subseteq L(Z,z)$. Then there is some $\xi' \in \xi\eta - L(Z,z)$. Let Δ be the smallest simplex of $\text{Link}(X,z)$ containing ξ' and $\omega_1, \cdots, \omega_k$ be its vertices. Then $\xi' \in B(\omega_j, \pi/2)$ for all $1 \leq j \leq k$. Since $L(Z,z)$ is a full subcomplex and $\xi' \notin L(Z,z)$, there exists some j, $1 \leq j \leq k$, with $\omega_j \notin L(Z,z)$. We may assume $\omega_1 \notin L(Z,z)$. By the definition of X_T, ω_1 corresponds to some edge $e \in E_i$ with $i \notin T$. It follows that $\xi, \eta \notin B(\omega_1, \pi/2)$. Then Lemma 2.2 implies that $\xi\eta \cap B(\omega_1, \pi/2)$ has length π, contradicting to the fact that $d_z(\xi,\eta) < \pi$.

When z is not a vertex, there is an obvious all right simplicial complex structure on $\text{Link}(X,z)$ where the vertices in $\text{Link}(X,z)$ are represented by geodesic segments parallel to the edges in X. In this case $L(Z,z)$ is still a full subcomplex of $\text{Link}(X,z)$ and the above argument applies.

It is clear that Z is a foldable cubical complex that is dimensionally homogeneous. Since X is geodesically complete, it has no boundary. It follows that Z also has no boundary. Z is locally convex in X implies Z has nonpositive curvature in the path metric. By Proposition 2.4 Z is geodesically complete and therefore is a FCC. □

Proposition 2.4 *Let Z be either a piecewise Euclidean complex that has nonpositive curvature, or a piecewise spherical complex that is locally* $\mathrm{CAT}(1)$. *Assume Z is locally finite and dimensionally homogeneous. Then Z is geodesically complete if and only if it has no boundary.*

Proof It is clear that if Z is geodesically complete then it has no boundary. To prove the other direction, we assume Z has no boundary. We first notice that Z is geodesically complete if and only if for any $z \in Z$ and any $\xi \in \mathrm{Link}(Z, z)$ there is some $\eta \in \mathrm{Link}(Z, z)$ with $d_z(\xi, \eta) \geq \pi$. The link $\mathrm{Link}(Z, z)$ is a finite piecewise spherical complex that is $\mathrm{CAT}(1)$, dimensionally homogeneous and has no boundary. We prove the following statement by induction on the dimension: Let Y be an n-dimensional finite piecewise spherical complex that is $\mathrm{CAT}(1)$, dimensionally homogeneous and has no boundary. Then for any $x \in Y$, there is some $y \in Y$ with $d(x, y) \geq \pi$.

When $n = 1$, Y is a finite graph and the claim is clear. Let $n = \dim Y \geq 2$ and suppose there is a point $x \in Y$ such that $d(x, y) < \pi$ for all $y \in Y$. Since Y is finite, there is some $y_0 \in Y$ with $d(x, y) \leq d(x, y_0)$ for all $y \in Y$. As Y is $\mathrm{CAT}(1)$ and $d(x, y_0) < \pi$, there is a unique minimizing geodesic $\sigma : [0, d(x, y_0)] \to Y$ from y_0 to x. Let $\xi \in \mathrm{Link}(Y, y_0)$ be the point represented by σ. Now $\mathrm{Link}(Y, y_0)$ has dimension $n - 1$ and the induction hypothesis implies that there is some $\eta \in \mathrm{Link}(Y, y_0)$ with $d_{y_0}(\xi, \eta) \geq \pi$, where d_{y_0} is the path metric on $\mathrm{Link}(Y, y_0)$. Hence σ can be extended to a geodesic $\tilde{\sigma} : [-\epsilon, d(x, y_0)] \to Y$ that contains y_0 in the interior. As Y is $\mathrm{CAT}(1)$, $\tilde{\sigma}$ is minimizing for small enough ϵ, contradicting to the choice of y_0. □

2.2 Graph of spaces decomposition

In this section we show that a FCC admits decomposition as a graph of spaces, as defined in [SW].

Let X be a FCC and $f : X \to C_0$ a fixed folding onto an n-cube. Then the set E of edges of X is a disjoint union $E = \amalg_{i=1}^{n} E_i$, see Section 2.1. For each i with $1 \leq i \leq n$, let $T_i = \{1, 2, \cdots, n\} - \{i\}$. Then the components of X_{T_i} are $(n-1)$-dimensional FCCs and are locally convex in X.

For each n-cube C of X, let $e \in E_i$ be an edge of C and $C_i \subset C$ the hyperplane in C containing the midpoint of e and perpendicular to e. It is clear that $C_i \subset C$ does not depend on e and is isometric to a $(n-1)$-dimensional unit Euclidean cube. Set $H_i = \cup C_i$, where C varies over all n-cubes of X. H_i is not connected in general. An argument similar to the one in Section 2.1 shows

that each component of H_i is locally convex in X. H_i has a natural FCC structure, where each C_i is a $(n-1)$-cube.

It is not hard to see that each component of $X - X_{T_i}$ is isometric to $Y \times (0,1)$, where Y is a component of H_i. Let $\{Y_1, \cdots, Y_k\}$ be the set of components of H_i. Then X can be obtained from X_{T_i} by attaching $Y_j \times [0,1]$ along $Y_j \times \{0\}$ and $Y_j \times \{1\}$, $1 \leq j \leq k$. That is, X has the structure of a graph of spaces [SW] for each i, $1 \leq i \leq n = \dim X$. Now we make it more precise. The base graph G_i is as follows. The vertex set $\{v_B\}$ of G_i is in one-to-one correspondence with the set $\{B\}$ of components of X_{T_i}, and the edge set $\{e_Y\}$ is in one-to-one correspondence with the set $\{Y\}$ of components of H_i. For each edge e_Y, consider the component $Y \times (0,1)$ of $X - X_{T_i}$ corresponding to Y. The closure of $Y \times (0,1)$ in X has nonempty intersection with one or two components of X_{T_i}. Let B_0, B_1 be these components of X_{T_i} (we may have $B_0 = B_1$). Then the edge e_Y connects the vertices v_{B_0} and v_{B_1}. The vertex space corresponding to v_B is the component B of X_{T_i} and the edge space corresponding to e_Y is the component Y of H_i.

We notice that the base graph G_i is connected: Let v_B, $v_{B'}$ be two vertices of G_i. Pick two vertices $v \in B, v' \in B'$ of X. Since X is connected, there are vertices $v_0 = v, v_1, \cdots, v_l = v'$ such that v_{j-1} and v_j are the endpoints of an edge e_j. Let B_j be the component of $X - X_{T_i}$ that contains v_j. If $e_j \notin E_i$, then $B_{j-1} = B_j$. On the other hand, if $e_j \in E_i$, then $v_{B_{j-1}}$ and v_{B_j} are connected by an edge in G_i.

We next describe the maps from edge spaces to vertex spaces. Let e_Y be an edge of G_i connecting v_{B_0} and v_{B_1}. We may assume that for each fixed $y \in Y$, (y,t) (where $(y,t) \in Y \times (0,1) \subset X$) converges to a point in B_0 as $t \to 0$ and to a point in B_1 as $t \to 1$. The maps $g_{e_Y,0} : Y \to B_0$ and $g_{e_Y,1} : Y \to B_1$ can be described as follows. Recall each $(n-1)$-cube of Y has the form C_i, where C is an n-cube of X and $C_i \subset C$ is the hyperplane of C containing the midpoint of some edge $e \in E_i$ of C and perpendicular to e. Clearly C has the decomposition $C_i \times [0,1]$. We may assume $C_i \times \{0\} \subset C$ is contained in B_0 and $C_i \times \{1\} \subset C$ is contained in B_1. Then the map $g_{e_Y,0} : Y \to B_0$ sends C_i to $C_i \times \{0\}$ via the identity map. Similarly for the map $g_{e_Y,1}$. Thus $g_{e_Y,0}$ and $g_{e_Y,1}$ are nondegenerate combinatorial maps between FCCs. Recall that two cubes in X either are disjoint or intersect in a single cube. It follows that the maps $g_{e_Y,0}$ and $g_{e_Y,1}$ are immersions, that is, they are locally injective.

Lemma 2.5 *The maps $g_{e_Y,0}$ and $g_{e_Y,1}$ are local isometric embeddings. In particular, they induce injective homomorphisms between fundamental groups.*

Proof We show that $g_{e_Y,0}$ is a local isometric embedding, the proof for $g_{e_Y,1}$ is similar. It suffices to show that $g_{e_Y,0} : Y \to B_0$ sends geodesics in Y to geodesics in B_0. Recall that Y is a component of H_i and the component of $X - X_{T_i}$ containing Y is isometric to $Y \times (0,1)$. Y can be identified with $Y \times \{\frac{1}{2}\}$. Let $\sigma : I \to Y$ be a geodesic. Then for each t, $0 < t < 1$, the map $\sigma_t : I \to X$ with $\sigma_t(s) = (\sigma(s), t)$ for $s \in I$ is a geodesic in X. Since X has nonpositive curvature, the limit map $\sigma_0 : I \to X$, $\sigma_0(s) := \lim_{t\to 0} \sigma_t(s)$ is also a geodesic in X(see p.121, Corollary 7.58 of [BH]). Now the lemma follows since $\sigma_0 = g_{e_Y,0} \circ \sigma$. □

Summarizing the above observations we have the following:

Proposition 2.6 *Let X be a* FCC *of dimension n. Then X admits the structure of a graph of spaces, where all the vertex and edge spaces are $(n-1)$-dimensional* FCC*s and the maps from edge spaces to vertex spaces are combinatorial immersions. In particular, $\pi_1(X)$ has a graph of groups decomposition.*

Let X, G_i be as above, $v_B \in G_i$ a vertex and $e_Y \subset G_i$ an edge incident to v_B. Suppose for each $y \in Y$, (y,t) converges to a point in B as $t \to 0$, and let $g_{e_Y,0} : Y \to B$ be the map from the edge space to the vertex space. For each vertex $w \in Y$, $\overline{\{w\} \times (0,1)}$ is an edge in X; we let e_w be the associated oriented edge which goes from 0 to 1. For any oriented edge e of X with initial point v, $\overrightarrow{e} \in \mathrm{Link}(X,v)$ denotes the point representing e.

Lemma 2.7 *In the above notation, the following two conditions are equivalent:*

(1) $g_{e_Y,0}$ *is not a covering map;*

(2) *there exists a vertex $w \in Y$ and an oriented edge $e \subset B$ with initial point $v := g_{e_Y,0}(w)$ such that $d_v(\overrightarrow{e}, \overrightarrow{e_w}) \geq \pi$.*

Proof We first notice that (2) is equivalent to the following condition:

(3) There is a vertex $w \in Y$ and an edge $e \subset B$ incident to $v = g_{e_Y,0}(w)$ such that no edge (in Y) incident to w is mapped to e.

So (2) clearly implies (1). Now assume $g_{e_Y,0}$ is not a covering map. If the image of $g_{e_Y,0}$ does not contain all the vertices of B, the connectedness of the 1-skeleton of B implies there is an edge $e \subset B$ with two endpoints v_1 and v_2 such that $v_1 \in \mathrm{image}(g_{e_Y,0})$ and $v_2 \notin \mathrm{image}(g_{e_Y,0})$. In this case no edge of Y is mapped to e and (3) holds. So we assume $\mathrm{image}(g_{e_Y,0})$ contains all the vertices of B. If (3) does not hold, then $\mathrm{image}(g_{e_Y,0}) = B$ and for each vertex $w \in Y$, $g_{e_Y,0}$ maps $star(w)$ isomorphically onto $star(v)$, where $v = g_{e_Y,0}(w)$. It follows that $g_{e_Y,0}$ is a covering map, contradicting to the assumption. □

2.3 Davis complexes of right-angled Coxeter groups

In this section we give examples of finite FCCs whose universal covers are Davis complexes of certain right-angled Coxeter groups.

Let S be a finite set, and $M = (m_{s,s'})_{s,s' \in S}$ a symmetric matrix such that $m_{s,s} = 1$ for $s \in S$ and $m_{s,s'} \in \{2, 3, \cdots, \} \cup \{\infty\}$ for $s \neq s' \in S$. The group W given by the presentation $W = < S | (ss')^{m_{s,s'}} = 1, s, s' \in S >$ is a Coxeter group, where $(ss')^{\infty} = 1$ means the relation is void. The Coxeter group W is a right-angled Coxeter group if for any $s \neq s' \in S$, $m_{s,s'} = 2$ or ∞.

Given any Coxeter group W, there is a locally finite cell complex D_W, the so called Davis complex of W such that W acts on D_W properly with compact quotient [D]. Moussong ([M], [D]) showed that there is a piecewise Euclidean metric on D_W that turns D_W into a CAT(0) space. The action of W on D_W preserves the piecewise Euclidean metric, that is, W acts on D_W as a group of isometries. When W is a right-angled Coxeter group, the Davis complex D_W is a cubical complex. Below we shall describe FCCs which are finite quotients of Davis complexes of certain right-angled Coxeter groups.

There is a one-to-one correspondence between right-angled Coxeter groups and finite flag complexes. Let W be a right-angled Coxeter group with standard generating set S. The nerve $N(W)$ of W is a simplicial complex with set of vertices S. For any nonempty subset $T \subset S$ there is a simplex in $N(W)$ with T as its vertex set if and only if $m_{t,t'} = 2$ for any $t \neq t' \in T$. $N(W)$ is clearly a flag complex. Conversely, let K be a finite flag complex with set of vertices S. Then we can define a right-angled Coxeter group with generating set S as follows: for $s \neq s'$, $m_{s,s'} = 2$ if there is an edge in K joining s and s', and $m_{s,s'} = \infty$ otherwise.

Let K be a finite flag complex with vertex set S. We shall construct a finite cubical complex $Y(K)$ whose vertex links are all isomorphic to K ([BH], p.212). Let V be a Euclidean space with dimension equal to $|S|$, the cardinality of S. Identify the standard basis $e_s (s \in S)$ with S and consider the unit cube $[0,1]^{|S|} \subset V$. For each nonempty subset $T \subset S$, let F_T be the face of $[0,1]^{|S|}$ spanned by the unit vectors e_t, $t \in T$. The cubical complex $Y(K)$ is the subcomplex of $[0,1]^{|S|}$ consisting of all faces parallel to F_T, for all nonempty subsets T of S that are the vertex sets of simplices of K. Notice that $Y(K)$ contains the 1-skeleton of the unit cube $[0,1]^{|S|} \subset V$. In particular, $Y(K)$ is connected.

Proposition 2.8 [BH] *Let K be a finite flag complex. Then $Y(K)$ is a connected finite cubical complex with nonpositive curvature all of whose vertex links are isomorphic to K.*

We subdivide $Y(K)$ by using the hyperplanes $x_s = 1/2$ $(s \in S)$ of the unit cube $[0,1]^{|S|}$. Let $X(K)$ be the obtained cubical complex. Then $X(K)$ is also a finite cubical complex of nonpositive curvature with some of its vertex links isomorphic to K.

A simplicial complex K is foldable if there is a simplicial map $f : K \to \Delta$ onto an n-simplex such that the restriction of f to each simplex is injective.

Corollary 2.9 *Let K be a finite flag complex. If K is foldable, dimensionally homogeneous and has no boundary, then $X(K)$ is a finite* FCC .

Proof It is not hard to see that $X(K)$ is dimensionally homogeneous and has no boundary. Proposition 2.4 implies that $X(K)$ is geodesically complete. We need to show that $X(K)$ is foldable. First we notice that the group $\mathbb{Z}_2^{|S|}$ acts on $X(K)$ as a group of isometries: the s-th factor $\mathbb{Z}_2$ acts as the orthogonal reflection about the hyperplane $x_s = 1/2$. Let $o \in X(K)$ be the origin of V and $star(o)$ the star of o in $X(K)$. Then the quotient of $X(K)$ by $\mathbb{Z}_2^{|S|}$ is isomorphic to $star(o)$. So we have a nondegenerate combinatorial map from $X(K)$ onto $star(o)$. Since the link of $X(K)$ at o is isomorphic to K and K is foldable, the star $star(o)$ can be folded according to the folding of K. The composition of these two maps is a folding of $X(K)$. □

Next we construct certain flag complexes that satisfy the assumptions in Corollary 2.9. Let $n \geq 2$. A standard n-sphere is the unit round n-sphere with an all right simplicial complex structure. A standard n-sphere has $n+1$ subcomplexes which are standard $(n-1)$-spheres; we call them equators. Let $\mathbb{S}^n$ be a standard n-sphere and E one of its equators. An all right simplicial complex is called a standard n-hemisphere if it is isomorphic to the closure of one of the components of $\mathbb{S}^n - E$; the subcomplex of the standard n-hemisphere corresponding to E is also called its equator. The unique point on a standard n-hemisphere that has distance $\pi/2$ from its equator is called its pole.

Let $\mathbb{S}^n$ be a standard n-sphere and $E_1, \cdots, E_{n+1}$ its equators. A *hemispherex* is an all right simplicial complex obtained from $\mathbb{S}^n$ by attaching a finite number of standard n-hemispheres along the equators of $\mathbb{S}^n$ such that for each $i, 1 \leq i \leq n+1$, there is at least one hemisphere attached along E_i. Here the

attaching is realized through isomorphisms between equators of $\mathbb{S}^n$ and those of the hemispheres. It is clear that a hemispherex H satisfies all the conditions in Corollary 2.9, so the corresponding $X(H)$ is a finite FCC. The universal cover of $X(H)$ is a subdivision of the Davis complex of the right-angled Coxeter group whose nerve is H.

Hemispherex was first introduced by Ballmann and Brin in [BBr2]. The Euclidean cone over a hemispherex is the first example of a higher rank CAT(0) space aside from Euclidean buildings and symmetric spaces. But such a space does not admit cocompact isometric actions. On the other hand, we shall see in Section 3.2 that if X is a finite FCC with some vertex link isomorphic to a hemispherex, then it contains closed rank one geodesics. In particular, $X(H)$ contains closed rank one geodesics if H is a hemispherex.

3 Existence of closed rank one geodesics

In this section we discuss the existence of closed rank one geodesics in a finite FCC. Throughout this section X denotes a finite FCC of dimension n, except in Section 3.4.

3.1 Vertex links and 1-skeleton

Recall that X is dimensionally homogeneous, geodesically complete and has nonpositive curvature. It follows that for each vertex v of X, the link $\text{Link}(X,v)$ is dimensionally homogeneous, has no boundary and is a flag complex.

Recall that the set of edges in X is a disjoint union: $E = \amalg_{i=1}^{n} E_i$. Let v be a vertex of X. For $1 \le i \le n$, let $V_{i,v}$ be the set of vertices in $\text{Link}(X,v)$ that correspond to edges in E_i. Since X has no boundary, $V_{i,v}$ contains at least two points.

Lemma 3.1 *Let $v \in X$ be a vertex and $1 \le i \le n$. If $V_{i,v}$ consists of exactly two points, then $\text{Link}(X,v)$ is the spherical join of $V_{i,v}$ and $L_{i,v}$, where $L_{i,v}$ is the subcomplex of $\text{Link}(X,v)$ consisting of all simplices of $\text{Link}(X,v)$ with vertices in $V_{j,v}$, $j \neq i$.*

Proof Denote by ξ_+, ξ_- the two vertices in $V_{i,v}$, and let $\eta \in V_{j,v}$ with $j \neq i$. Since $\text{Link}(X,v)$ is a flag complex it suffices to show that there are edges $\eta\xi_+$ and $\eta\xi_-$ in $\text{Link}(X,v)$.

Since $\mathrm{Link}(X,v)$ is dimensionally homogeneous, there is a $(n-1)$-simplex Δ_1 of $\mathrm{Link}(X,v)$ containing η. Δ_1 contains exactly one vertex from $V_{i,v}$. Without loss of generality we may assume it is ξ_+ (thus there is an edge $\eta\xi_+$). Let $\Delta_1' \subset \Delta_1$ be the $(n-2)$-face disjoint from ξ_+. Since $\mathrm{Link}(X,v)$ has no boundary, there is a $(n-1)$-simplex $\Delta_2 \neq \Delta_1$ containing Δ_1'. The assumption that $V_{i,v}$ consists of exactly two points implies ξ_- is a vertex of Δ_2. In particular there is an edge $\eta\xi_-$. □

For any vertex v of X and $1 \le i \le n$, let $X_{i,v}$ be the component of $X_{\{i\}}$ that contains v.

Corollary 3.2 *Let $v \in X$ be a vertex and $1 \le i \le n$. If there are $\xi \in V_{j,v}$, $\eta \in V_{i,v}$ with $j \neq i$ and $d_v(\xi,\eta) \ge \pi$, then $X_{i,v}$ is not a circle.*

Proof By Lemma 3.1, $V_{i,v}$ contains at least three points and therefore there are at least three edges of $X_{i,v}$ incident to v. □

For an oriented edge e, $\bar{e}$ denotes the same edge with the opposite orientation and $t(e)$ denotes the terminal point of e. It is not hard to prove the following lemma.

Lemma 3.3 *Let Γ be a connected finite graph such that the valence of each vertex is at least two. Assume Γ is not homeomorphic to a circle. Then for any two oriented edges e_1, e_2 in Γ, there is a geodesic c from $t(e_1)$ to $t(e_2)$ such that $\bar{e}_2 * c * e_1$ is also a geodesic. In particular, for any oriented edge e of Γ, there is a geodesic loop c based at $t(e)$ such that $\bar{e} * c * e$ is also a geodesic.*

3.2 Closed rank one geodesics

Let $v \in X$ be a vertex and $\xi \in \mathrm{Link}(X,v)$. Then ξ lies in the interior of a simplex Δ of $\mathrm{Link}(X,v)$. Define $T(\xi) = \{i : V_{i,v} \cap \Delta \neq \phi\}$. Then $T(\xi) \subset \{1,2,\cdots,n\}$. We say $T(\xi)$ is the type of ξ. For an oriented edge e of X with initial point v, define $P_e = \{\xi \in \mathrm{Link}(X,v) : d_v(\xi, \overrightarrow{e}) = \pi/2\}$.

Lemma 3.4 *Let e be an oriented edge of X. Then there is an isometry $D_e : P_e \to P_{\bar{e}}$ such that ξ and $D_e(\xi)$ have the same type for any $\xi \in P_e$.*

Proof Let i be the index such that the geometric edge of e lies in E_i, m the midpoint of e and Y the component of H_i containing m. Denote by v and w the initial and terminal points of e, and B_0 and B_1 the components of X_{T_i} containing v and w respectively. Since $g_{e_Y,0} : Y \to B_0$ and $g_{e_Y,1} : Y \to B_1$ are local isometric embeddings and B_0, B_1 are locally convex in X, the induced maps $h_0 : \mathrm{Link}(Y, m) \to \mathrm{Link}(B_0, v) \subset \mathrm{Link}(X, v)$ and $h_1 : \mathrm{Link}(Y, m) \to \mathrm{Link}(B_1, w) \subset \mathrm{Link}(X, w)$ are isometric embeddings. It is not hard to check that the images of these maps are P_e and $P_{\bar{e}}$ respectively. Set $D_e = h_1 \circ {h_0}^{-1}$. Then $D_e : P_e \to P_{\bar{e}}$ is an isometry. Since h_0 and h_1 clearly preserve type, D_e also preserves type. □

Corollary 3.5 *Let X be a* FCC *, $v \in X$ a vertex and e_1, e_2, e_3 three oriented edges with initial point v. Suppose $d_v(\overrightarrow{e_1}, \overrightarrow{e_3}) = d_v(\overrightarrow{e_2}, \overrightarrow{e_3}) = \pi/2$. Then $d_v(\overrightarrow{e_1}, \overrightarrow{e_2}) = \pi$ if and only if $d_w(D_{e_3}(\overrightarrow{e_1}), D_{e_3}(\overrightarrow{e_2})) = \pi$, where w is the terminal point of e_3.*

Proof It follows easily from the fact that $D_{e_3} : P_{e_3} \to P_{\bar{e}_3}$ is an isometry. □

We say a geodesic in a metric space with nonpositive curvature has rank one if its lifts in the universal cover have rank one.

Proposition 3.6 *Let $c \subset X$ be a closed geodesic that is contained in the* 1*-skeleton. If for each i, c contains at least one edge from E_i, then c is a closed rank one geodesic.*

Proof Let $\pi : \tilde{X} \to X$ be the universal cover of X, $\tilde{c}$ a lift of c to $\tilde{X}$ and $\tilde{E}_i = \pi^{-1}(E_i)$ for $1 \le i \le n$. Note $\tilde{X}$ is also a FCC and the set $\tilde{E}$ of edges of $\tilde{X}$ has the decomposition into different colors $\tilde{E} = \amalg_{i=1}^{n} \tilde{E}_i$.

Assume the proposition is false. Then there is half flatplane H bounding $\tilde{c}$, that is, $H = \mathrm{image}(f)$ where f is an isometric embedding $f : \mathbb{R} \times [0, \infty) \to \tilde{X}$ with $f(\mathbb{R} \times \{0\}) = \tilde{c}$. For any vertex $v \in \tilde{c}$, H determines a unique point $\xi_v \in \mathrm{Link}(\tilde{X}, v)$ with the following property: for any oriented edge $e \subset \tilde{c}$ with initial point v and terminal point w, $\xi_v \in P_e$ and $D_e(\xi_v) = \xi_w$. By Lemma 3.4, all ξ_v have the same type $T \subset \{1, 2, \cdots, n\}$. Let $i \in T$. By assumption there is an oriented edge $e \subset \tilde{c}$ with $e \in \tilde{E}_i$. Let v be the initial point of e. Since $\mathrm{Link}(\tilde{X}, v)$ is an all right spherical complex and $\overrightarrow{e} \in \mathrm{Link}(\tilde{X}, v)$ is a vertex, $\xi_v \in P_e$ implies $d_v(\overrightarrow{e}, \xi) = \pi/2$ for any vertex ξ of the smallest simplex of $\mathrm{Link}(\tilde{X}, v)$ containing ξ_v. Since $T(\xi_v) = T \ni i$, the definition of type implies there is an oriented edge $e_i \in \tilde{E}_i$ with $d_v(\overrightarrow{e}, \overrightarrow{e_i}) = \pi/2$. This contradicts to the facts that $e \in \tilde{E}_i$ and $\tilde{X}_{i,v}$ is locally convex in $\tilde{X}$. □

Let $v \in X$ be a vertex. We define a relation $\sim_v$ on the set $\{1, 2, \cdots, n\}$ as follows: $i \sim_v j$ if and only if there are oriented edges $e_1 \in E_i$, $e_2 \in E_j$ with initial point v such that $d_v(\overrightarrow{e_1}, \overrightarrow{e_2}) \geq \pi$.

Proposition 3.7 *Let X be a finite* FCC *of dimension n. For a fixed vertex $v \in X$, if $T \subset \{1, 2, \cdots, n\}$ is an equivalence class with respect to the equivalence relation generated by $\sim_v$, then there is a closed geodesic in the* 1*-skeleton of X that contains at least one edge from E_i for each $i \in T$.*

Proof By the definition of the equivalence relation, we may assume $T = \{i_1, i_2, \cdots, i_m\}$ such that for each t, $2 \leq t \leq m$, there is some j, $1 \leq j < t$ with $i_t \sim_v i_j$. Now we prove the following claim by induction on k, $1 \leq k \leq m$: there is a closed geodesic c_k in the 1-skeleton of X such that for each $j \leq k$, c_k contains an edge belonging to E_{i_j} and incident to v.

The claim is clear for $k = 1$. Now let $k \geq 2$ and assume the claim has been established for $k-1$. By the above paragraph we have $i_k \sim_v i_j$ for some $j < k$. Hence there are oriented edges $e'_j \in E_{i_j}$, $e_k \in E_{i_k}$ with initial point v such that $d_v(\overrightarrow{e'_j}, \overrightarrow{e_k}) \geq \pi$. Corollary 3.2 implies $X_{i_j,v}$ and $X_{i_k,v}$ are not circles. By induction hypothesis, there is a closed geodesic c_{k-1} in the 1-skeleton of X containing an oriented edge $e_j \in E_{i_j}$ with initial point v. Let v_1, v_2, v_3 be the terminal points of the edges e_k, e'_j, e_j respectively. By Lemma 3.3, there is a geodesic loop $\tilde{c}_1$ in $X_{i_k,v}$ based at v_1 such that $c'_1 = \bar{e}_k * \tilde{c}_1 * e_k$ is also a geodesic in $X_{i_k,v}$. Similarly there are geodesic loops $\tilde{c}_2$ and $\tilde{c}_3$ in $X_{i_j,v}$ based at v_2 and v_3 respectively such that $c'_2 = \bar{e}'_j * \tilde{c}_2 * e'_j$ and $c'_3 = \bar{e}_j * \tilde{c}_3 * e_j$ are geodesics in $X_{i_j,v}$. Proposition 2.3 implies c'_1, c'_2 and c'_3 are geodesics in X. We reparametrize the closed geodesic c_{k-1} so that it starts from v with initial segment e_j. Define $c_k = c'_2 * c'_3 * c_{k-1} * c'_2 * c'_1$ if $e'_j \neq e_j$ and $c_k = c'_1 * c'_2 * c_{k-1}$ if $e'_j = e_j$. Now it is easy to check that c_k is a closed geodesic with the required property. □

The following corollary follows immediately from Propositions 3.6 and 3.7.

Corollary 3.8 *Let X be a finite* FCC *of dimension n and $v \in X$ a vertex. If $\{1, 2, \cdots, n\}$ is a single equivalence class with respect to the equivalence relation generated by $\sim_v$, then there is a closed rank one geodesic contained in the* 1*-skeleton of X.*

Let H be a hemispherex with central sphere $\mathbb{S}^n$, and $E_1, \cdots, E_{n+1}$ the equators of $\mathbb{S}^n$. For each i, $1 \leq i \leq n+1$, let H_i be a fixed hemisphere of H that is

attached to $\mathbb{S}^n$ along E_i. Denote the pole of H_i by p_i. Then the distance between p_i and p_j is π for $i \neq j$. Now let X be a FCC and $v \in X$ a vertex with $\text{Link}(X, v) = H$. Then there are oriented edges e_i, $1 \leq i \leq n+1$ with initial point v and $\overrightarrow{e_i} = p_i$. It is clear that e_i and e_j have different colors for $i \neq j$. It follows that the assumption in the above corollary is satisfied if the vertex link $\text{Link}(X, v)$ is a hemispherex. Thus we have:

Corollary 3.9 *Let X be a finite* FCC *of dimension n. Suppose there is a vertex $v \in X$ such that* $\text{Link}(X, v)$ *is a hemispherex, then X has a closed rank one geodesic contained in the 1-skeleton.*

Proposition 3.10 *Let X be a finite* FCC *. Suppose there is a vertex $v \in X$ and $\xi \in V_{i,v}$, $\eta \in V_{j,v}$ with $i \neq j$ and $d_v(\xi, \eta) > \pi$, then X contains a closed rank one geodesic in the 1-skeleton.*

Proof Let e_1 and e_2 be the two oriented edges with initial point v that give rise to ξ and η respectively. Since $d_v(\xi, \eta) > \pi$, Corollary 3.2 implies $X_{i,v}$ and $X_{j,v}$ are not circles. Lemma 3.3 then implies there are geodesic loops $c_1 \subset X_{i,v}$ and $c_2 \subset X_{j,v}$ based at $t(e_1)$ and $t(e_2)$ respectively such that $c_1' := \bar{e}_1 * c_1 * e_1$ and $c_2' := \bar{e}_2 * c_2 * e_2$ are geodesics in $X_{i,v}$ and $X_{j,v}$ respectively. Since by Proposition 2.3 $X_{i,v}$ and $X_{j,v}$ are locally convex in X, c_1' and c_2' are geodesics in X. Let $c = c_2' * c_1'$. Since $d_v(\xi, \eta) > \pi$, it is clear that c is a closed rank one geodesic. □

3.3 A splitting criterion

Let Y be a CAT(0) space and $Z_1, Z_2 \subset Y$ be two closed, convex subsets. We say Z_1, Z_2 are *parallel* if for some $a \geq 0$ there is an isometric embedding $f : Z_1 \times [0, a] \to Y$ such that $f(Z_1 \times \{0\}) = Z_1$ and $f(Z_1 \times \{a\}) = Z_2$. For any closed convex subset $Z \subset Y$ of a CAT(0) space Y, let P_Z be the union of all closed convex subsets that are parallel to Z. When Z is geodesically complete, P_Z is closed, convex and isometrically splits $Z \times C$, where $C \subset Y$ is closed and convex ([BBr2], p.6).

Proposition 3.11 *Let X be a* FCC *of dimension n. Suppose $\{1, 2, \cdots, n\}$ is the disjoint union of nonempty subsets T, S with the following property: for any vertex $v \in X$, and any two edges incident to v, $e_i \in E_i$, $e_j \in E_j$ with $i \in T$, $j \in S$, there is a square containing e_i and e_j in the boundary. Then the universal cover $\tilde{X}$ of X is isometric to the product of two* CAT(0) FCCs *.*

Proof For any vertex $v \in \tilde{X}$, let $\tilde{X}_{T,v}$ be the component of $\tilde{X}_T$ that contains v. We claim for any edge e of $\tilde{X}$ with endpoints v and w, $\tilde{X}_{T,v}$ and $\tilde{X}_{T,w}$ are parallel.

We may assume $e \in E_i$ for some $i \in S$, otherwise $\tilde{X}_{T,v} = \tilde{X}_{T,w}$. For any $k \geq 0$ we inductively define a subcomplex $\tilde{X}_{T,v}(k)$ of $\tilde{X}_{T,v}$: $\tilde{X}_{T,v}(0) = \{v\}$, for $k \geq 1$, $\tilde{X}_{T,v}(k)$ is the union of $\tilde{X}_{T,v}(k-1)$ and all the cubes in $\tilde{X}_{T,v}$ that have nonempty intersection with $\tilde{X}_{T,v}(k-1)$. Similarly one can define $\tilde{X}_{T,w}(k)$. We also define subcomplexes $\tilde{X}_{T,e}(k)$ of $\tilde{X}$: $\tilde{X}_{T,e}(0) = e$, for $k \geq 1$, $\tilde{X}_{T,e}(k)$ is the union of $\tilde{X}_{T,e}(k-1)$ and all the cubes whose edges are in $E_i \cup (\cup_{j \in T} E_j)$ and whose intersections with $\tilde{X}_{T,e}(k-1)$ contain edges from E_i. Set $\tilde{X}_{T,e} = \cup_{k \geq 0} \tilde{X}_{T,e}(k)$.

Since $\tilde{X}$ is a CAT(0) cubical complex, the vertex links of $\tilde{X}$ are flag complexes. Our assumption then implies that for any $(m-1)$ $(m \leq n)$ cube C in $\tilde{X}_{T,v}(1)$ or $\tilde{X}_{T,w}(1)$, there is a unique m-cube in $\tilde{X}$ that contains both e and C. It follows that $\tilde{X}_{T,e}(1)$ contains $\tilde{X}_{T,v}(1)$ and $\tilde{X}_{T,w}(1)$ and there is an isomorphism

$$f_{e,1} : \tilde{X}_{T,v}(1) \times [0,1] \to \tilde{X}_{T,e}(1)$$

such that $f_{e,1|\tilde{X}_{T,v}(1) \times \{0\}}$ is the identity map and $f_{e,1}(\tilde{X}_{T,v}(1) \times \{1\}) = \tilde{X}_{T,w}(1)$. Now $\tilde{X}_{T,v}(k) = \cup_{v'} \tilde{X}_{T,v'}(1)$ and $\tilde{X}_{T,e}(k) = \cup_{e'} \tilde{X}_{T,e'}(1)$, where v' varies over all vertices in $\tilde{X}_{T,v}(k-1)$ and $e' \subset \tilde{X}_{T,e}(k-1)$ varies over all edges from E_i. Notice all the maps $f_{e',1}$ are compatible for $e' \subset \tilde{X}_{T,e}(k-1)$ from E_i. It follows that for each k there is an isomorphism $f_{e,k} : \tilde{X}_{T,v}(k) \times [0,1] \to \tilde{X}_{T,e}(k)$ such that $f_{e,k|\tilde{X}_{T,v}(k) \times \{0\}}$ is the identity map, $f_{e,k}(\tilde{X}_{T,v}(k) \times \{1\}) = \tilde{X}_{T,w}(k)$ and $f_{e,k}$ agrees with $f_{e,k-1}$ when restricted to $\tilde{X}_{T,v}(k-1) \times [0,1]$. The union of all these isomorphisms $f_{e,k}$ defines an isomorphism $f_e : \tilde{X}_{T,v} \times [0,1] \to \tilde{X}_{T,e}$ such that $f_{e|\tilde{X}_{T,v} \times \{0\}}$ is the identity map onto $\tilde{X}_{T,v}$ and $f_e(\tilde{X}_{T,v} \times \{1\}) = \tilde{X}_{T,w}$. It follows that $\tilde{X}_{T,v}$ and $\tilde{X}_{T,w}$ are parallel.

Fix a vertex $v_0 \in \tilde{X}$ and let P_T be the parallel set of $\tilde{X}_{T,v_0}$. Note $\tilde{X}_{T,v_0}$ is closed, convex and geodesically complete. It follows that P_T isometrically splits $P_T = \tilde{X}_{T,v_0} \times Y$ where $Y \subset \tilde{X}$ is a closed convex subset and for $y \in Y$, $\tilde{X}_{T,v_0} \times \{y\}$ is parallel to $\tilde{X}_{T,v_0}$. By the claim we have established, all vertices of $\tilde{X}$ lie in P_T. Since $\tilde{X}$ is the convex hull of all its vertices we see $\tilde{X} = P_T$ splits. □

Proposition 3.11 has previously been established in dimension 2 (Theorem 1.10 on p.36 of [W] and Theorem 10.2 in [BW]).

Let X be a FCC. By Proposition 2.6 X has the structure of a graph of spaces, where all the vertex and edge spaces are FCCs and the maps from edge spaces

to vertex spaces are combinatorial immersions. The following corollary follows from Lemma 2.7 and Proposition 3.11.

Corollary 3.12 *Let X be a* FCC *with a graph of spaces decomposition as in Proposition 2.6. If all the maps from edge spaces to vertex spaces are covering maps, then the universal cover of X is isometric to the product of a simplicial tree and a $(n-1)$-dimensional* CAT(0) FCC .

3.4 Rank rigidity in low dimensions

In this section we discuss the rank rigidity problem for finite FCCs with dimension ≤ 3. A 1-dimensional finite FCC X is a finite graph and each of its vertices is incident to at least two edges; X clearly contains closed geodesics and all the geodesics in X have rank one. The claim in dimension 2 follows easily from Corollary 3.8 and Proposition 3.11.

Theorem 3.13 *Let X be a finite* FCC *of dimension* 3 *with universal cover $\tilde{X}$.*

(1) *If $\tilde{X}$ has higher rank, then $\tilde{X}$ is isometric to the product of two* CAT(0) FCC*s .*

(2) *If $\tilde{X}$ has rank one, then there is a closed rank one geodesic in the* 1*-skeleton of X.*

Proof Suppose that $\tilde{X}$ does not split as a product. Recall the decomposition of the set of edges into different colors: $E = E_1 \amalg E_2 \amalg E_3$. We shall call the edges in E_1, E_2, E_3 blue, green and red edges respectively. By Proposition 3.10 and Proposition 3.11 we may assume the following: for any two oriented edges e_1, e_2 with the same initial point v but different colors, $d_v(\vec{e_1}, \vec{e_2}) \leq \pi$ holds; there exist two oriented edges e_1, e_2 with different colors (say blue and green respectively) and the same initial point v such that $d_v(\vec{e_1}, \vec{e_2}) = \pi$.

Consider the graph of spaces decomposition of X where the vertex spaces are components of $X_{\{1,2\}}$. Let G_3 be the base graph. By Corollary 3.12 at least one of the maps from the edge spaces to vertex spaces is not a covering map. Recall the base graph G_3 is connected. Let $k \geq 0$ be the smallest integer with the following property: there are two vertices v_B, $v_{B'}$ of G_3 at distance k apart such that

(1) B contains an oriented blue edge e_1 and an oriented green edge e_2 with the same initial point $v \in X$ such that $d_v(\vec{e_1}, \vec{e_2}) = \pi$;

(2) there is an edge $e_Y \subset G_3$ incident to $v_{B'}$ such that the map from the edge space Y to the vertex space B' is not a covering map.

We claim $k=0$. Assume $k \geq 1$. Let $B_0 = B, B_1, \cdots, B_k = B'$ be a sequence of components of $X_{\{1,2\}}$ such that v_{B_i} and $v_{B_{i+1}}$ $(0 \leq i \leq k-1)$ are adjacent vertices in G_3. Since $k \geq 1$, the map from the edge space of $v_B v_{B_1}$ to the vertex space B is a covering map. Lemma 2.7 implies that there is a red edge e with one endpoint v and the other endpoint w in B_1 such that e is perpendicular to both e_1 and e_2. Corollary 3.5 implies that there exist an oriented blue edge e_3 and an oriented green edge e_4 with initial point w such that $d_w(\overrightarrow{e_3}, \overrightarrow{e_4}) = \pi$. Note $e_3, e_4 \subset B_1$ and the distance from v_{B_1} to $v_{B'}$ in G_3 is $k-1$, contradicting to the definition of k. Therefore $k=0$ and $B = B'$. By Lemma 2.7 there is a vertex $v' \in B$ and oriented edges e_r (red), $e_b \subset B$ with initial point v' such that $d_{v'}(\overrightarrow{e_r}, \overrightarrow{e_b}) = \pi$. We may assume e_b is a blue edge.

By Corollary 3.8 we may assume the following: for any vertex $v \in B$, if there are oriented blue edge e_1 and red edge e_2 with initial point v such that $d_v(\overrightarrow{e_1}, \overrightarrow{e_2}) = \pi$, then all green edges incident to v are perpendicular to all blue and red edges incident to v.

Recall that B is a finite FCC of dimension 2. We consider the graph of spaces decomposition of B where the vertex spaces consist of blue edges. Let G be the connected base graph. Lemma 2.7 and the condition (1) above imply that not all maps from edge spaces to vertex spaces are covering maps. Let $l \geq 0$ be the smallest integer with the following property: there are two vertices v_C and $v_{C'}$ of G at distance l apart such that

(1) there is a vertex $v' \in C$, an oriented red edge e_r and an oriented blue edge e_b with initial point v' such that $d_{v'}(\overrightarrow{e_r}, \overrightarrow{e_b}) = \pi$;

(2) there is an edge $e_Y \subset G$ incident to $v_{C'}$ such that the map from the edge space Y to the vertex space C' is not a covering map.

Now the preceding paragraph and a similar argument as above show that $l=0$ and $C = C'$.

There is a vertex $v'' \in C$, an oriented blue edge $e_{b'}$ and an oriented green edge e_g with initial point v'' such that $d_{v''}(\overrightarrow{e_{b'}}, \overrightarrow{e_g}) = \pi$. Corollary 3.2 implies $C = X_{1,v''}$ is not a circle. By Lemma 3.3 there is a geodesic $c \subset C$ from v' to v'' which starts with e_b and ends with $\bar{e}_{b'}$. Similarly there are geodesic loops c_1 and c_2 based at $t(e_r)$ and $t(e_g)$ respectively such that $c_1' = \bar{e}_r * c_1 * e_r$ and $c_2' = \bar{e}_g * c_2 * e_g$ are also geodesics. Set $c' = c_1' * \bar{c} * c_2' * c$. Then c' is a closed geodesic that contains blue, green and red edges. By Proposition 3.6 c' is a rank one geodesic. □

Theorem 3.13 (1) also follows from a theorem of Ballmann and Brin [BBr2]: they proved that if a 3-dimensional, geodesically complete and piecewise Euclidean polyhedra Y is CAT(0), has higher rank and admits a cocompact and properly discontinuous group of isometries, then Y either isometrically splits or is a thick Euclidean building of type $\tilde{A}_3$ or $\tilde{B}_3$. A FCC certainly can not be an Euclidean building of type $\tilde{A}_3$ or $\tilde{B}_3$. The main point of Theorem 3.13 is the existence of closed rank one geodesics in rank one finite FCCs. Our proof of Theorem 3.13 is independent of the proof in [BBr2].

4 Tits alternative for foldable cubical complexes

In this section we give a short proof of the Tits alternative for the fundamental group of a finite FCC. Ballmann and Swiatkowski have a slightly more general result [BSw].

Theorem 4.1 *Let X be a finite* FCC *. Then any subgroup of $\pi_1(X)$ either contains a free group of rank two or is virtually free abelian.*

Proof We induct on the dimension of X. If $\dim X = 1$, then X is a finite graph and the Theorem clearly holds. Let $n = \dim X$ and H a subgroup of $\pi_1(X)$. By Proposition 2.6 X admits a graph of spaces decomposition where all the vertex and edge spaces are $(n-1)$-dimensional finite FCCs. It follows that $\pi_1(X)$ admits a graph of groups decomposition and acts on the associated Bass-Serre tree T. As a subgroup of $\pi_1(X)$, H also acts on T. By [PV], H contains a free group of rank two unless one of the following happens: H fixes a point in T, H stabilizes a complete geodesic in T, or H fixes a point in $\partial_\infty T$. We need to consider these three exceptional cases.

First assume H fixes a point in T. Then H fixes a vertex of T and is a subgroup of a conjugate of some vertex group of the graph of groups decomposition for $\pi_1(X)$. We have observed that such a vertex group is the fundamental group of a finite FCC with dimension $n-1$. By induction hypothesis the claim on H holds.

Now assume H stabilizes a complete geodesic c in T. Since T is a simplicial tree, by taking an index two subgroup we may assume H acts on c as translations and so there is an exact sequence $1 \to N \to H \to \mathbb{Z} \to 1$. Thus $N \subset H$ has a fixed point in $c \subset T$. By the previous paragraph N contains a free group of rank two or is virtually free abelian. We may assume N is virtually free abelian. It implies that H is virtually solvable. The claim on H follows since

any virtually solvable subgroup of a group acting properly and cocompactly by isometries on a CAT(0) space is virtually free abelian ([BH], p.249). The case when H fixes a point in $\partial_\infty T$ can be handled similarly. □

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